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Covariant-tensor method for quantum groups and applications I: $SU(2)_q$

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Abstract. A covariant-tensor method for $SU(2)_q$ is described. This tensor method is used to calculate q-deformed Clebsch–Gordan coefficients. The connection with covariant oscillators and irreducible tensor operators is established. This approach can be extended to other quantum groups.

1. Introduction

In recent years there has been considerable interest in q-deformations of Lie algebras (quantum groups) [1] and their applications in physics [2]. The main goal of these applications is a generalization of the concept of symmetry. The properties of quantum groups are similar to those of classical Lie groups with q not being a root of unity. However, it is still not clear to what extent the familiar tensor methods, used in the representation theory of Lie algebras, are applicable to the case of q-deformations.

Different types of tensor calculus for $SU(2)_q$ were proposed and applied in references [3–7]. However, no simple covariant-tensor calculus for $SU(n)_q$ was presented. In this paper we propose a simple covariant-tensor method for $SU(2)_q$ which can be extended to the general $SU(n)_q$. Details for $SU(n)_q$ and especially for $SU(3)_q$ will be published separately.

The plan of the paper is the following. In section 2 we recall the basics of the $SU(2)_q$ algebra, its fundamental representation and invariants. In section 3 we construct the general $SU(2)_q$-covariant tensors and invariants. In section 4 we apply this tensor method to calculate q-deformed Clebsch–Gordan coefficients and in section 5 we demonstrate their symmetries. We point out that this method is simpler than that used in previous calculations [5, 8] and can be generalized to other quantum groups. Finally, in section 6 we connect covariant tensors with covariant q-oscillators and construct unit irreducible tensor operators.

2. $SU(2)_q$-algebra, its fundamental representation and invariants

Let us recall that three generators of $SU(2)_q$ obey the following commutation relations [1] (we take q real)

$$[J^0, J^\pm] = \pm J^\pm \tag{2.1}$$

$$[J^+, J^-] = [2J^0]_q = \frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}}.$$

The coproduct Δ; $SU(2)_q \to SU(2)_q \otimes SU(2)_q$ is defined as

$$\begin{aligned}\Delta(J^\pm) &= J^\pm \otimes q^{J^0} + q^{-J^0} \otimes J^\pm \\ \Delta(J^0) &= J^0 \otimes 1 + 1 \otimes J^0.\end{aligned} \tag{2.2}$$

Let V_2 be a two-dimensional space spanned by the basis $|e_a\rangle$, $a = 1, 2$, and $|v\rangle = \Sigma_a |e_a\rangle v_a \in V_2$. The $SU(2)_q$ generators J^k ($k = \pm, 0$) act as

$$\begin{aligned}J^k|e_a\rangle &= \sum_b (J^k)_{ba}|e_b\rangle \\ J^k|v\rangle &= \sum_{a,b} (J^k)_{ba}|e_b\rangle v_a \\ &= \sum_b |e_b\rangle (J^k V)_b \\ &= \sum_b |e_b\rangle v'_b.\end{aligned} \tag{2.3}$$

In the fundamental representation of $SU(2)_q$ the generator J^ks are ordinary 2×2 Pauli matrices.

Let $(V_2)^*$ be a dual space with the basis $\langle e_a| = (|e_a\rangle)^+$ and $\langle v| = (|v\rangle)^+ = \Sigma_a v_a^* \langle e_a|$. The dual basis is orthonormal, i.e. $\langle e_a|e_b\rangle = \delta_{ab}$. We note that the components of the vector $|v\rangle$, v_a, (or v_a^* of $\langle v|$) are not defined as real or complex numbers. Their algebraic properties follow from $SU(2)_q$-invariance requirements. Here we identify (for the spin $j = \frac{1}{2}$)

$$\begin{aligned}|e_a\rangle &= |\tfrac{1}{2}, m_a\rangle \\ \langle e_a| &= \langle \tfrac{1}{2}, m_a|\end{aligned} \qquad m_a = \pm\tfrac{1}{2} \tag{2.4}$$

and the matrix elements of the generators J^k are

$$\begin{aligned}\langle e_a|J^0|e_a\rangle &= m_a \\ \langle e_1|J^+|e_2\rangle &= \langle e_2|J^-|e_1\rangle = 1.\end{aligned} \tag{2.5}$$

We define a scalar product as $\langle u|v\rangle = \Sigma_a u_a^* v_a$ and the norm as $\langle v|v\rangle = \Sigma_a v_a^* v_a$. This scalar product (and the norm) are not $SU(2)_q$-invariant. Instead, the quantity

$$\langle v|q^{-J^0}|v\rangle \tag{2.6}$$

is invariant under the action of the coproduct (2.2) in the following sense:

$$\begin{aligned}\Delta(J^\pm)\langle v|q^{-J^0}|v\rangle &= (J^\pm\langle v|)|v\rangle + (q^{-J^0}\langle v|)J^\pm q^{-J^0}|v\rangle \\ &= -\langle v|J^\pm|v\rangle + \langle v|J^\pm|v\rangle = 0 \\ \Delta(J^0)\langle v|q^{-J^0}|v\rangle &= (J^0\langle v|)q^{-J^0}|v\rangle + \langle v|J^0 q^{-J^0}|v\rangle \\ &= -\langle v|J^0 q^{-J^0}|v\rangle + \langle v|J^0 q^{-J^0}|v\rangle \\ &= 0.\end{aligned} \tag{2.7}$$

The quadratic forms

$$\sum_a u_a^* q^{-J^0} v_a = \sum_a u_a^* q^{-m_a} v_a \tag{2.8}$$

and

$$\sum_a v_a q^{m_a} u_a^* \tag{2.9}$$

are $SU(2)_q$-invariant. Note that the first quadratic form (2.8) can be written as $\langle u|q^{-J_0}|v\rangle$.

If we demand $\Sigma_a v_a^* q^{-m_a} v_a = \Sigma_a v_a q^{m_a} v_a^*$, it follows that $v_1^* v_1 = q v_1 v_1^*$ and $v_2^* v_2 = q^{-1} v_2 v_2^*$.

In addition to the $\langle u|q^{-J_0}|v\rangle$-invariant form we consider another form

$$\varepsilon_{ab}|e_a\rangle|e_b\rangle \tag{2.10}$$

with

$$\begin{aligned} \varepsilon_{ab} &= \begin{pmatrix} 0 & q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix} \\ \varepsilon_{ab}\varepsilon_{bc} &= -\delta_{ac} \\ (\varepsilon_{\bar a \bar b})_q &= (\varepsilon_{ba})_q = -(\varepsilon_{ab})_{q^{-1}} \end{aligned} \tag{2.11}$$

where $\bar 1 = 2$ and $\bar 2 = 1$.

Note that the q-antisymmetric combination $v_a v_b \varepsilon_{ab}$ is $SU(2)_q$-invariant, showing that v_a and v_b do not commute. Instead, they q-commute, i.e. $v_2 v_1 = q v_1 v_2$.

3. General $SU(2)_q$-tensors and invariants

Let us consider the tensor-product space $(V_2)^{\otimes k} = V_2 \otimes \ldots \otimes V_2$ with the basis $|e_{a_1}\rangle \otimes \ldots \otimes |e_{a_k}\rangle$, $a_1, \ldots a_k = 1, 2$. Then we write an element of the tensor space $(V_2)^{\otimes k}$ as tensor $|T\rangle$ of the form

$$\begin{aligned} |T\rangle &= |e_{a_1}\rangle \ldots |e_{a_k}\rangle T^{a_1} \ldots T^{a_k} \\ &= |e_{a_1} \ldots e_{a_k}\rangle T^{a_1 \ldots a_k}. \end{aligned} \tag{3.1}$$

We have the following proposition:

> The tensor $|T\rangle$ transforms under the $SU(2)_q$ algebra as an irreducible representation of spin $j = k/2$ if and only if $T^2 T^1 = q T^1 T^2$.
>
> Let us assume $T^2 T^1 = q T^1 T^2$. Then

$$\begin{aligned} |T_{j=k/2}\rangle &= |e_{a_1} \ldots e_{a_k}\rangle T^{a_1 \ldots a_k} \\ &= \sum_{m=-j}^{+j} |jm\rangle T^{jm}. \end{aligned} \tag{3.2}$$

The vectors $|jm\rangle$ span the space V_{2j+1} of the irreducible representation with spin j. From $T^2 T^1 = q T^1 T^2$ it follows that

$$T^{a_1 \ldots a_k} = q^{\chi(a_1 \ldots a_k)} : T^{a_1 \ldots a_k} : \tag{3.3}$$

where $:T:$ means the normal order of indices (1s on the left of 2s), i.e. $T^{11\ldots122\ldots2}$ and index 1 (2) appears n_1 (n_2) times, respectively. χ is the number of inversions with respect to the normal order. Hence

$$\begin{aligned} |\mathrm{jm}\rangle &= |\mathrm{e}_{\{a_1 \ldots a_k\}}\rangle \\ &= \frac{1}{\sqrt{f}} q^{-M/2} \sum_{\mathrm{perm}(a_1 \ldots a_k)} q^{\chi(a_1 \ldots a_k)} |e_{a_1 \ldots a_k}\rangle \end{aligned} \tag{3.4}$$

where the curly bracket $\{a_1 \ldots a_k\}$ denote the q-symmetrization. The summation runs over all the allowed permutations of the fixed set of indices (n_1 1s and n_2 2s) and

$$M = n_1 n_2 = (j+m)(j-m)$$
$$j = \tfrac{1}{2}(n_1+n_2) \qquad m = \tfrac{1}{2}(n_1-n_2) \tag{3.5}$$
$$f = \binom{2j}{j+m}_q = \frac{[2j]_q!}{[j+m]_q![j-m]_q!}.$$

The important relation is

$$f = q^{-M} \sum_{\mathrm{perm}(a_1 \ldots a_k)} q^{2x(a_1 \ldots a_k)}. \tag{3.6}$$

From equation (3.4) and the definition of the coproduct $\Delta(J^\pm)$ (2.2) we can reproduce

$$\Delta(J^\pm)|jm\rangle = \sqrt{[j \mp m]_q [j \pm m + 1]_q}\, |jm \pm 1\rangle$$
$$\Delta(J^0)|jm\rangle = m|jm\rangle. \tag{3.7}$$

From (3.2) and (3.4) we immediately obtain the relation between T^{jm} and the components of $T^{a_1 \ldots a_k}$:

$$T^{jm} = q^{M/2}\sqrt{f}\colon T^{a_1 \ldots a_k}\colon$$
$$T^{j-m} = q^{M/2}\sqrt{f}\colon T^{\bar{a}_1 \ldots \bar{a}_k}\colon \tag{3.8}$$

where $\bar{1} = 2$, $\bar{2} = 1$ and $T^{j-m} = (T^{jm})_{n_1 \Leftrightarrow n_2}$.

In the dual space $(V_2^{\otimes k})^*$ we define

$$\langle e_{a_k \ldots a_1}| = (|e_{a_1 \ldots a_k}\rangle)^+$$
$$\langle e_{a_k \ldots a_1}|e_{b_1 \ldots b_k}\rangle = \delta_{a_1 b_1} \ldots \delta_{a_k b_k} \tag{3.9}$$

and in the dual space $(V_{2j+1})^*$ we define

$$\langle jm| = (|jm\rangle)^+ = \langle e_{\{a_k \ldots a_1\}}|$$
$$= \frac{1}{\sqrt{f}} q^{-M/2} \sum_{\mathrm{perm}(a_1 \ldots a_k)} q^{x(a_1 \ldots a_k)} (|e_{a_1 \ldots a_k}\rangle)^+$$
$$= \frac{1}{\sqrt{f}} q^{-M/2} \sum_{\mathrm{perm}(a_1 \ldots a_k)} q^{x(a_1 \ldots a_k)} \langle e_{a_k \ldots a_1}|. \tag{3.10}$$

As a consequence of equations (3.4), (3.6) and (3.9) we obtain

$$\langle jm_1 | jm_2\rangle = \frac{1}{f} q^{-M} \sum_{\mathrm{perm}(a_1 \ldots a_k)} q^{2x(a_1 \ldots a_k)} \delta_{m_1 m_2}$$
$$= \delta_{m_1 m_2}. \tag{3.11}$$

The $SU(2)_q$-invariant quantity built up of the tensors $\langle T|$ and $|U\rangle$ of spin $j = k/2$ is

$$I = \langle T|q^{-J^0}|U\rangle = (T^{a_k \ldots a_1})^* q^{-J^0} U^{a_1 \ldots a_k}$$
$$= \sum_{m=-j}^{+j} (T^{jm})^* U^{jm} q^{-m}.$$

The second type of the $SU(2)_q$-invariant quantity built up of the tensors $|T\rangle$ and $|U\rangle$ of spin $j=k/2$ is

$$I' = T^{a_k \dots a_1} U^{b_1 \dots b_k} \varepsilon_{a_1 b_1} \varepsilon_{a_2 b_2} \dots \varepsilon_{a_k b_k} \tag{3.13}$$

with ε_{ab} given in (2.11). Of course, $T^a T^b \varepsilon_{ab} = 0$ if T^a and T^b q-commute.

Furthermore, using equation (3.3) we can also write

$$\begin{aligned} I &= q^{\chi(s)} (T^{a_k \dots a_1})^* q^{-J^0} U^{s(a_1 \dots a_k)} \\ I' &= q^{\chi(s)} T^{a_k \dots a_1} U^{s(b_1 \dots b_k)} \varepsilon_{a_1 b_1} \dots \varepsilon_{a_k b_k} \end{aligned} \tag{3.14}$$

where $s \in S_k$ is a fixed permutation of the indices $a_1 \dots a_k$ and $\chi(s) = \chi(a_1 \dots a_k) - \chi(s(a_1 \dots a_k))$ is the number of inversions with respect to the $(a_1 \dots a_k)$ order.

4. q-Clebsch–Gordan coefficients

Here we present a new simple method for calculating the q-deformed Clebsch–Gordan (C-G) coefficients. It can be immediately extended and applied to $SU(n)_q$ and other quantum groups. This method is a consequence of the previously described tensor method and construction of invariants.

Our notation is

$$|JM\rangle = \sum_{m_1, m_2} \langle j_1 m_1 \; j_2 m_2 | JM \rangle_q |j_1 m_1\rangle |j_2 m_2\rangle. \tag{4.1}$$

For $q \in \boldsymbol{R}$, C-G coefficients are real

$$\langle j_1 m_1 \; j_2 m_2 | JM \rangle_q^* = \langle j_1 m_1 \; j_2 m_2 | JM \rangle_q \tag{4.2}$$

and

$$\langle j_1 \; m_1 j_2 m_2 | JM \rangle_q = \langle JM | j_1 m_1 \; j_2 m_2 \rangle_q. \tag{4.3}$$

Using the tensor notation $|jm\rangle = |e_{\{a_1 \dots a_k\}}\rangle$ ((3.4), (3.9) and (3.10)), we first calculate C-G coefficient for $j_1 \otimes j_2 \to j_1 + j_2$:

$$\begin{aligned} \langle j_1 + j_2 \; m_1 + m_2 | j_1 m_1 \; j_2 m_2 \rangle_q &= \langle e_{\{b_l \dots b_1, a_k \dots a_1\}} | e_{\{a_1 \dots a_k\}} e_{\{b_1 \dots b_l\}} \rangle \\ &= \langle e_{\{b,a\}} | e_{\{a\}} e_{\{b\}} \rangle \\ &= \frac{1}{\sqrt{f_1 f_2 f_3}} q^{-\frac{1}{2}(M_1 + M_2 + M_3)} \sum_{\text{perm}(a),(b)} q^{\chi(a) + \chi(b) + \chi(a,b)} \\ &= \sqrt{\frac{f_1 f_2}{f_3}} q^{\frac{1}{2}(M_1 + M_2 - M_3)} q^{(j_1 - m_1)(j_2 + m_2)} \\ &= \sqrt{\frac{f_1 f_2}{f_3}} q^{j_1 m_2 - j_2 m_1} \end{aligned} \tag{4.4}$$

where we have used

$$\chi(a, b) = \chi(a) + \chi(b) + (j_1 - m_1)(j_2 + m_2) \tag{4.5}$$

and equation (3.6) together with the abbreviations

$$
\begin{aligned}
&k=2j_1 \qquad l=2j_2 \qquad j_3=j_1+j_2 \qquad m_3=m_1+m_2 \\
&M_i=(j_i+m_i)(j_i-m_i) \qquad f_i=\binom{2j_i}{j_i+m_i}_q \\
&\frac{f_1 f_2}{f_3}=\frac{[2j_1]_q![2j_2]_q![j_3+m_3]q![j_3-m_3]_q!}{[j_1+m_1]_q![j_1-m_1]_q![j_2+m_2]_q![j_2-m_2]_q![2j_3]_q!}.
\end{aligned}
\tag{4.6}
$$

The main observation is that any C-G coefficient $\langle j_1m_1\ j_2m_2|JM\rangle$ can be written in the form (4.4). Namely, the C-G coefficient $\langle j_1m_1\ j_2m_2|JM\rangle$ is projection of the state $\langle j_1m_1|\otimes\langle j_2m_2|=\langle e_{\{a_1\ldots a_{2j_1}\}}e_{\{b_1\ldots b_{2j_2}\}}|$ from the tensor product space $V^*_{2j_1+1}\otimes V^*_{2j_2+1}$ to the state $|JM\rangle=|e_{\{a_1\ldots[a_{2j_1-n+1}[\ldots[a_{2j_1},b_1].4..]b_n]\ldots b_{2j_2}\}}\rangle$ (with the appropriate symmetry of $2j_1+2j_2$ indices) in the space $V_{2J+1}\subset V_{2j_1+1}\otimes V_{2j_2+1}$. Here, the square brackets $[\ldots]$ denote q-antisymmetrization and $n=2j=j_1+j_2-J$. Furthermore, the state $|e_{[a_1[a_2\ldots[a_n,b_n]\ldots b_2]b_1]}\rangle\infty\varepsilon_{a_nb_n}\ldots\varepsilon_{a_1b_1}$ transforms as a singlet, i.e. it is invariant under the coproduct action in the tensor product space $V_n\otimes V_n$. Hence, using the equation (3.4), we can write

$$
\begin{aligned}
\langle j_1m_1\ j_2m_2|JM\rangle_q
&=\mathcal{N}\sum_{\substack{\text{perm}\,(a,b)\\(c,d)}}\langle e_{\{a,b\}}e_{\{c,d\}}|e_{\{a,d\}}\rangle\cdot(\varepsilon_{(b,c)})_n \\
&=\mathcal{N}\frac{q^{-\frac{1}{2}(M_1+M_2+M_J)}}{\sqrt{f_1f_2f_J}}\sum_{\substack{\text{perm}\,(a,b)\\(c,d)}}q^{x(a,b)+x(c,d)+x(a,d)}(\varepsilon_{(b,c)})_n
\end{aligned}
\tag{4.7}
$$

where the length of b (c) is $n=j_1+j_2-J$, $(\varepsilon_{(b,c)})_n=\varepsilon_{b_1c_1}\ldots\varepsilon_{b_nc_n}$ and

$$
\mathcal{N}=\left(\frac{[2j_1]_q![2j_2]_q![2J+1]_q!}{[j_1+j_2-J]_q![j_1-j_2+J]_q![-j_1+j_2+J]_q![j_1+j_2+J+1]_q!}\right)^{1/2}.
\tag{4.8}
$$

Expression (4.7) is efficient for practical calculation of C-G coefficients (see appendix).

We also present a simple derivation of the standard expression for q-C-G coefficients [5].

Using the decomposition

$$
\begin{aligned}
\langle j_1m_1| &=\sum_{m=-j}^{+j}\langle j_1m_1|j_1-j\ m_1-m\ jm\rangle_q\langle j_1-j\ m_1-m|\langle jm| \\
\langle j_2m_2| &=\sum_{m=-j}^{+j}\langle j_2m_2|j-m\ j_2-j\ m_2+m\rangle_q\langle j-m|\langle j_2-j\ m_2+m| \\
|JM\rangle &=\sum_{m=-j}^{+j}\langle j_1-j\ m_1-m\ j_2-j\ m_2+m|JM\rangle_q|j_1-j\ m_1-m\rangle|j_2-j\ m_2+m\rangle
\end{aligned}
\tag{4.9}
$$

we immediately write

$$
\begin{aligned}
\langle j_1m_1\ j_2m_2|JM\rangle_q=N\sum_{m=-j}^{+j}&\langle j_1m_1|j_1-j\ m_1-m\ jm\rangle_q \\
&\times\langle j_2m_2|j-m\ j_2-j\ m_2+m\rangle_q\langle jm\ j-m|00\rangle_q \\
&\times\langle j_1-j\ m_1-m\ j_2-j\ m_2+m|JM\rangle_q
\end{aligned}
\tag{4.10}
$$

where N is the norm depending on j_1, j_2 and J. Three of the four C-G coefficients appearing on the right-hand side have the simple form (4.4). The fourth coefficient $\langle jm\ j-m|00\rangle$ also has a simple form. Namely, for $n=2j$ we have

$$\begin{aligned}\langle jm\ j-m|00\rangle_q &= \frac{1}{\sqrt{[n+1]_q}}\varepsilon_{a_1b_1}\ldots\varepsilon_{a_nb_n}\\ &= \frac{1}{\sqrt{[2j+1]_q}}q^{\frac{1}{2}n_1}(-q^{-\frac{1}{2}})^{n_2}\\ &= (-1)^{j-m}\frac{1}{\sqrt{[2j+1]_q}}q^{m}.\end{aligned} \tag{4.11}$$

The denominator $[2j+1]^{1/2}$ comes from the orthonormality condition.

Finally, inserting equations (4.4) and (4.11) into equation (4.10) we find

$$\begin{aligned}&\langle j_1m_1\ j_2m_2|JM\rangle_q\\ &\quad= N\sum_{m=-j}^{+j}\frac{(-1)^{j-m}}{\sqrt{[2j+1]_q}}q^{j_1m_2-j_2m_1}\\ &\qquad\times q^{m(2J+2j+1)}\frac{\begin{pmatrix}2j\\ j+m\end{pmatrix}_q\begin{pmatrix}2j_1-2j\\ j_1-j+m_1-m\end{pmatrix}_q\begin{pmatrix}2j_2-2j\\ j_2-j+m_2+m\end{pmatrix}_q}{\sqrt{\begin{pmatrix}2J\\ J+M\end{pmatrix}_q\begin{pmatrix}2j_1\\ j_1+m_1\end{pmatrix}_q\begin{pmatrix}2j_2\\ j_2+m_2\end{pmatrix}_q}}\end{aligned} \tag{4.12}$$

with $j_1+j_2-j=J+j$. This result agrees with the result found by Ruegg [5] if the normalization factor N is taken as

$$N=\left\{\frac{[2j_1]_q![2j_2]_q![2J+1]_q![j_1+j_2-J+1]_q}{[j_1+j_2-J]_q![j_1-j_2+J]_q![-j_1+j_2+J]_q![j_1+j_2+J+1]_q!}\right\}^{1/2}. \tag{4.13}$$

We point out that our tensor method is simple and can be easily applied to $SU(n)_q$ for $n\geqslant 3$. We also mention that it can be applied to multiparameter quantum groups. For example, it can be shown [9] that C-G coefficients for the two-parameter $SU(2)_{p,q}$ [10] depend effectively on one parameter only.

5. Symmetry relations

For completeness we rederive the known symmetry relations for q-C-G coefficients and q-$3-j$ symbols. From equation (4.4) immediately follow symmetry relations

$$\begin{aligned}\langle j_1-m_1\ j_2-m_2|j_1+j_2\ -m_1-m_2\rangle_q &= \langle j_2m_2\ j_1m_1|j_1+j_2\ \ m_1+m_2\rangle_q\\ &= \langle j_1m_1\ j_2m_2|j_1+j_2\ \ m_1+m_2\rangle_{q^{-1}}\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}&\langle j_1-m_1\ j_1+j_2\ \ m_1+m_2|j_2m_2\rangle_q\\ &\quad=(-1)^{j_1+m_1}q^{-m_1}\sqrt{\frac{[2j_2+1]_q}{[2j_1+2j_2+1]_q}}\langle j_1m_1\ j_2m_2|j_1+j_2\ \ m_1+m_2\rangle_q.\end{aligned} \tag{5.2}$$

Furthermore, from equation (4.11) we have

$$\begin{aligned}&\langle j-m\ jm|00\rangle_q=(-1)^{2j}\langle jm\ j-m|00\rangle_{q^{-1}}\\ &\langle jm\ 00|jm\rangle_q=1.\end{aligned} \tag{5.3}$$

The symmetry relations (5.1)-(5.3) are sufficient to derive the symmetries of the general C-G coefficients. From equation (4.10) we obtain

$$\langle j_1-m_1\ j_2-m_2|J-M\rangle_q=\langle j_2m_2\ j_1m_1|JM\rangle_q$$
$$=(-1)^{j_1+j_2-J}\langle j_1m_1\ j_2m_2|JM\rangle_{q^{-1}} \tag{5.4}$$

and

$$\langle j_1-m_1\ JM|j_2m_2\rangle_q=(-1)^{J-j_2+m_1}q^{-m_1}\sqrt{\frac{[2j_2+1]_q}{[2J+1]_q}}\langle j_1m_1\ j_2m_2|JM\rangle_q. \tag{5.5}$$

(One can deduce this directly from (4.7).)

We can define the q-deformed $3-j$ symbol as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q = q^{\frac{1}{3}(m_2-m_1)}\frac{(-1)^{j_1-j_2-m3}}{\sqrt{[2j_3+1]_q}}\langle j_1m_1\ j_2m_2|j_3-m_3\rangle_q \tag{5.6}$$

where the additional factor $q^{\frac{1}{3}(m_2-m_1)}$ comes from the requirement that symmetry relations for the $(3-j)_q$ coefficients should not contain explicit q-factors:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}_q = \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix}_q = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_{q^{-1}} \tag{5.7}$$

and that the $(3-j)_q$ coefficients are invariant under cyclic permutations.

Note that the $SU(2)_q$ invariant, built up of the three states $|j_1m_1\rangle$, $|j_2m_2\rangle$ and $|j_3m_3\rangle$, is

$$\sum_{m_1,m_2,m_3}\langle j_3-m_3\ j_3m_3|00\rangle_q\langle j_1m_1\ j_2m_2|j_3-m_3\rangle_q|j_1m_1\rangle|j_2m_2\rangle|j_3m_3\rangle$$
$$=\sum_{m_1,m_2,m_3}q^{\frac{2}{3}(m_1-m_3)}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q|j_1m_1\rangle|j_2m_2\rangle|j_3m_3\rangle$$
$$=\sum_{m_1,m_2,m_3}N_{123}(\varepsilon_{(b,c)})_{k_1}(\varepsilon_{(d,e)})_{k_2}(\varepsilon_{(a,f)})_{k_3}|e_{\{a,b\}}\rangle|e_{\{c,d\}}\rangle|e_{\{e,f\}}\rangle. \tag{5.8}$$

Now we identify

$$\langle j_1m_1\ j_2m_2|j_3-m_3\rangle_q\langle j_3-m_3\ j_3m_3|00\rangle_q=q^{\frac{2}{3}(m_1-m_3)}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q$$
$$=N_{123}(\varepsilon_{(b,c)})_{k_1}(\varepsilon_{(d,e)})_{k_2}(\varepsilon_{(a,f)})_{k_3} \tag{5.9}$$

where, for example, $(\varepsilon_{(b,c)})_k=\varepsilon_{b_1c_1}\ldots\varepsilon_{b_kc_k}$ with

$$k_1=j_1+j_2-j_3 \qquad k_2=-j_1+j_2+j_3 \qquad k_3=j_1-j_2+j_3 \tag{5.10}$$

and N_{123} is the normalization factor fully symmetric in indices (123). Equation (5.9) represents the connection with the tensor notation used (see (5.7)).

6. Covariant q-oscillators and irreducible tensor operators

Let us define the q-bosonic operators a_i and a^+ $(i=1,2)$ such that $|e_i\rangle=a_i^+|0,0\rangle_F$ and $\langle e_i|={}_F\langle 0,0|a_i$, where $|0,0\rangle_F$ denotes the (Fock) vacuum state invariant under $SU(2)_q$. Hence, a_1^+ and a_2^+ are covariant operators transforming as an $SU(2)_q$ doublet. Therefore, analogously as in equation (3.2), they q-commute

$$a_2^+a_1^+=qa_1^+a_2^+. \tag{6.1}$$

Furthermore, we define the projector $P_{(j=k/2)}$ from the tensor space $(V_2)^{\otimes k}$ to the totally q-symmetric space carrying an irreducible representation of spin $j=k/2$

$$P_{(j=k/2)}|e_{i_1\dots i_k}\rangle = \frac{1}{\sqrt{[k]_q!}}a^+_{i_1}\dots a^+_{i_k}|0,0\rangle_F$$
$$= \frac{1}{\sqrt{[k]_q!}}q^{\chi(i_1\dots i_k)}(a_1^+)^{n_1}(a_2^+)^{n_2}|0,0\rangle_F. \tag{6.2}$$

We find from equation (3.4) that

$$|jm\rangle = q^{M/2}\frac{(a_1^+)^{n_1}(a_2^+)^{n_2}}{\sqrt{[n_1]_q![n_2]_q!}}|0,0\rangle_F$$
$$j=\tfrac{1}{2}(n_1+n_2) \qquad m=\tfrac{1}{2}(n_1-n_2). \tag{6.3}$$

We define the number operators N_i and N as

$$N_i|jm\rangle = N_i|n_1,n_2\rangle = n_i|n_1,n_2\rangle$$
$$N=N_1+N_2 \qquad [N,N_i]=0 \qquad [N_1,N_2]=0$$
$$[N_i,a_j^+]=\delta_{ij}a_i^+ \qquad [N_i,a_j]=-\delta_{ij}a_i$$
$$[N,a_i^+]=a_i^+ \qquad [N,a_i]=-a_i. \tag{6.4}$$

The action of a_i^+ and a_i on the basis vectors $|jm\rangle$ is

$$a_1^+|jm\rangle = q^{-\frac{1}{2}n_2}\sqrt{[n_1+1]_q}\,|j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle$$
$$a_2^+|jm\rangle = q^{\frac{1}{2}n_1}\sqrt{[n_2+1]_q}\,|j+\tfrac{1}{2},m-\tfrac{1}{2}\rangle$$
$$a_1|jm\rangle = q^{-\frac{1}{2}n_2}\sqrt{[n_1]_q}\,|j-\tfrac{1}{2},m-\tfrac{1}{2}\rangle$$
$$a_2|jm\rangle = q^{\frac{1}{2}n_1}\sqrt{[n_2]_q}\,|j-\tfrac{1}{2},m+\tfrac{1}{2}\rangle. \tag{6.5}$$

The commutation relations between a_i and a_j^+ follow immediately:

$$a_2^+a_1^+ = qa_1^+a_2^+ \qquad a_2a_1=q^{-1}a_1a_2$$
$$a_2a_1^+=a_1^+a_2 \qquad a_1a_2^+=a_2^+a_1 \tag{6.6}$$

and

$$a_1a_1^+=q^{-N_2}[N_1+1]_q \qquad a_1^+a_1=q^{-N_2}[N_1]_q$$
$$a_2a_2^+=q^{+N_1}[N_2+1]_q \qquad a_2^+a_2=q^{+N_1}[N_2]_q$$
$$H=a_1^+a_1+a_2^+a_2=[N]_q. \tag{6.7}$$

Then

$$a_1a_1^+-qa_1^+a_1=q^{-N}$$
$$a_2a_2^+-q^{-1}a_2^+a_2=q^{+N} \tag{6.8}$$

and

$$a_1a_1^+-q^{-1}a_1^+a_1=q^{2J^0}$$
$$a_2a_2^+-qa_2^+a_2=q^{2J^0}. \tag{6.9}$$

The generators $J^{\pm}$ and J^0 can be represented as

$$\begin{aligned} J^+ &= q^{-J^0+1/2}a_1^+a_2 \\ J^- &= q^{-J^0-1/2}a_2^+a_1 \\ 2J^0 &= N_1 - N_2 \\ [J^+, J^-] &= [2J^0]_q = [N_1-N_2]_q \\ [N, J^{\pm}] &= [N, J^0] = 0. \end{aligned} \tag{6.10}$$

We point out that the oscillator operators a_i and a_i^+ are covariant since the corresponding tensors $|e_{\{i_1 \ldots i_k\}}\rangle$, equation (3.4), are covariant and irreducible by construction.

We note that the covariant q-Bose operators a, a^+ (6.1) are the same as in [6], where they were constructed using the Wigner $D^{(j)}$-functions. A different set of covariant operators was constructed in [7]. Other constructions [11] are non-covariant in the sense that operators do not transform as $SU(2)_q$ doublet. In the non-covariant approach one has to solve an additional problem of constructing covariant, irreducible tensor operators [12].

The definition of the irreducible tensor operators of $SU(2)_q$ is

$$\begin{aligned} &(J^{\pm}T_{km} - q^{-m}T_{km}J^{\pm})q^{-J^0} = \sqrt{[k\mp m]_q[k\pm m+1]_q}\,T_{km\pm 1} \\ &[J^0, T_{km}] = mT_{km} \\ &|jm\rangle = T_{jm}|0,0\rangle_F. \end{aligned} \tag{6.11}$$

According to equations (6.1)–(6.3) we define a unit tensor operator as

$$T_{jm} = q^{\frac{1}{2}n_1n_2}\frac{(a_1^+)^{n_1}(a_2^+)^{n_2}}{\sqrt{[n_1]_q![n_2]_q!}} \tag{6.12}$$

which is covariant and irreducible by construction and satisfies the requirements (6.11) automatically. Note that $(T_{km})^+$ transforms as contravariant tensor. One can define the tensor

$$V_{k\mu} = (-1)^{k-\mu}q^{\mu}T^+_{k-\mu} \tag{6.13}$$

which transforms as covariant, irreducible tensor. In tensor notation we have

$$V^+_{\{i_1\ldots i_k\}} = \varepsilon_{i_1j_1}\ldots\varepsilon_{i_kj_k}T_{\{j_1\ldots j_k\}} = (-1)^{n_2}q^{\frac{1}{2}(n_1-n_2)}T_{k-\mu}. \tag{6.14}$$

For completeness, we present relations between the Biedenharn operators b_i, b_i^+ [11], t_i, t_i^+ [7] and a_i, a_i^+ of the present paper:

$$\begin{aligned} b_1 &= q^{-N_2-\frac{1}{2}N_1}t_1 = q^{\frac{1}{2}N_2}a_1 \\ b_2 &= q^{-\frac{1}{2}N_2}t_2 = q^{-\frac{1}{2}N_1}a_2 \\ b_1^+ &= t_1^+q^{-N_2-\frac{1}{2}N_1} = a_1^+q^{\frac{1}{2}N_2} \\ b_2^+ &= t_2^+q^{-\frac{1}{2}N_2} = a_2^+q^{-\frac{1}{2}N_1}. \end{aligned} \tag{6.15}$$

We point out that the general covariant oscillators (e.g. t_i, t_i^+ and a_i, a_i^+) are characterized by the anionic type q-commutation relation (6.1). Actually, equation (6.1) is a consequence of underlying braid group symmetry and can be also obtained from the $SU(2)_q$ $\check{R}$-matrix [7].

Finally, we give the Borel–Weil realization

$$a_i^+ \equiv x_i \qquad a_i \equiv D_i \qquad i = 1, 2 \tag{6.16}$$

which is covariant automatically. The commutation relations are

$$
\begin{aligned}
&x_2x_1=qx_1x_2 \qquad D_2D_1=q^{-1}D_1D_2\\
&D_1x_1=qx_1D_1+q^{-N} \qquad D_2X_2=q^{-1}x_2D_2+q^N\\
&[D_i,x_j]=0 \qquad i\neq j
\end{aligned}
\tag{6.17}
$$

or

$$
\begin{aligned}
&D_1x_1=q^{-1}x_1D_1+q^{2J^0}\\
&D_2x_2=qx_2D_2+q^{2J^0}
\end{aligned}
\tag{6.18}
$$

where

$$
\begin{aligned}
&N_i=x_i\partial_i\\
&\partial_i=\partial/\partial x_i.
\end{aligned}
\tag{6.19}
$$

It follows that

$$
\begin{aligned}
&D_ix_i^n=[n]_qx_i^{n-1}\\
&D_1=\frac{1}{x_1}[x_1\partial_1]_qq^{-x_2\partial_2}\\
&D_2=\frac{1}{x_2}[x_2\partial_2]_qq^{x_1\partial_1}.
\end{aligned}
\tag{6.20}
$$

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Appendix

We demonstrate usefulness of the equation (4.7) for practical calculations. Using equations (4.5) and (4.11) we write

$$
\begin{aligned}
&\chi(a,b)=\chi(a)+\chi(b)+n_2(a)n_1(b)\\
&\chi(c,d)=\chi(c)+\chi(d)+n_2(c)n_1(d)\\
&\chi(a,d)=\chi(a)+\chi(d)+n_2(a)n_1(d)\\
&\chi(b)=\chi(c)\\
&(\varepsilon_{(b,c)})_n=(-1)^{n_2(b)}q^{\frac{1}{2}(n_1(b)-n_2(b))}
\end{aligned}
\tag{A.1}
$$

where

$$
\begin{aligned}
&n=2j=j_1+j_2-J\\
&n_1(b)=n_2(c)=j+m\\
&n_2(b)=n_1(c)=j-m\\
&n_1(a)=j_1-j+m_1-m\\
&n_2(a)=j_1-j-m_1+m\\
&n_1(d)=j_2-j+m_2+m\\
&n_2(d)=j_2-j-m_2-m.
\end{aligned}
\tag{A.2}
$$

After inserting equation (3.6) into equation (4.7), we immediately obtain the final result, equation (4.12):

$$N\frac{q^{-\frac{1}{2}(M_1+M_2+M_J)}}{(f_1f_2f_J)^{1/2}}\sum_{n_1(b)=0}^{2j}\sum_{\text{perm}(a)}\sum_{\text{perm}(b)}\sum_{\text{perm}(d)}$$

$$\times q^{n_2(a)n_1(b)+n_1(b)n_1(d)+n_2(a)n_1(d)}q^{2\chi(a)+2\chi(b)+2\chi(d)}(\varepsilon_{(b,c)})_{2j}$$

$$=N\frac{q^{-\frac{1}{2}(M_1+M_2+M_J)}}{\sqrt{f_1f_2f_J}}\sum_{m=-j}^{+j}q^{n_2(a)n_1(b)+n_1(b)n_1(d)+n_2(a)n_1(d)}$$

$$\times f_af_bf_dq^{n_1(a)n_2(a)+n_1(b)n_2(b)+n_1(d)n_2(d)}(\varepsilon_{(b,c)})_{2j}$$

$$=N\sum_{m=-j}^{+j}(-1)^{j-m}q^{j_1m_2-j_2m_1}q^{m(2J+2j+1)}\frac{f_af_bf_d}{\sqrt{f_1f_2f_J}}. \tag{A.3}$$

We extend this simple calculation of the $\mathrm{SU}(2)_q$ C-G coefficients to the $\mathrm{SU}(N)_q$ quantum groups in the forthcoming paper.

References

[1] Drinfeld V G 1986 Quantum groups, ICM Proceedings, Berkely, p 978
Jimbo M 1986 *Lett. Mat. Phys.* **11** 247
Kirillov A N and Reshetikhin N Yu 1989 *Advanced Series in Mathematical Physics* vol 7 ed V G Kac (Singapore; World Scientific) p 285

[2] Alvarez Gaume L, Gomez C and Sierra G 1990 *Nucl. Phys.* B **330** 347
Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523
Bonatsos D, Faessler A, Raychev P P, Roussev R P and Smirnov Yu F 1992 *J. Phys. A: Math. Gen.* **25** 3275
Chang Z 1992 *J. Phys. A: Math. Gen.* **25** L781

[3] Schlieker M and Scholl M 1990 *Z. Phys.* C **47** 625
Song X C 1992 *J. Phys. A: Math. Gen.* **25** 2929

[4] Nomura M 1991 *J. Phys. Soc. Japan* **60** 789

[5] Ruegg H 1990 *J. Math. Phys.* **31** 1085

[6] Nomura M 1991 *J. Phys. Soc. Japan* **60** 3260

[7] Hadjiivanov L K, Paunov R R and Todorov I T 1992 *J. Math. Phys.* **33** 1379

[8] Groza V A, Kachurik I I and Klimyk A U 1990 *J. Math. Phys.* **31** 2769
Nomura M 1990 *J. Phys. Soc. Japan* **59** 439

[9] Meljanac S and Mileković M in preparation

[10] Burdik Ć and Hlavaty L 1991 *J. Phys. A: Math. Gen.* **24** L165

[11] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
Song X C 1990 *J. Phys. A: Math. Gen.* **23** L1155

[12] Pan F 1991 *J. Phys. A: Math. Gen.* **24** L803
Rittenberg V and Scheunert M 1992 *J. Math. Phys.* **33** 436